

Errors and Uncertainties

All measurements are to some degree uncertain. It is good scientific practice to **include an estimate of the uncertainty of every measurement you make**. The uncertainty of a single measurement—such as, for example, the diameter of a cylinder—comes from multiple sources, which fall into two categories: *random uncertainties* and *systematic errors*:

- There may be **actual variations** of the quantity being measured, e.g. the diameter of a cylinder may actually be different in different places. This random uncertainty can be reduced by making measurements at different places and taking the average. The standard deviation of the measurements (see below) is a measure of the uncertainty.
- The **measuring device** itself may be in error. The systematic error thus introduced will not lie equally on both sides of the true value, so averaging a large number of readings is no help. To reduce such errors, we calibrate the measuring device. In the case of the micrometer caliper, for example, we measure the zero error (the reading when the jaws are closed) and the readings on selected precision gauges that are about as large as the cylinder to be measured.
- The **measuring technique** may be a source of systematic error. When used to measure the diameter of a cylinder, the micrometer will always measure the largest diameter between its jaws; hence if there are small bumps or depressions on the cylinder, the average of a large number of measurements will not give the true average diameter but a quantity somewhat larger. This error can be reduced by making the jaws of the caliper smaller in cross section.
- The measuring technique may also be another source of random uncertainty. For example, the reading of the micrometer caliper may vary because one can't close it with the same force every time, and one's estimate of the fraction of the smallest division varies from trial to trial. Making multiple measurements also reduces the uncertainty from this source

Characterization of Uncertainties

The experimenter cannot immediately determine the cause of the variation of a set of measurements. A simpler task is to characterize this variation.

Consider the data below. Column 1 represents 10 readings of the diameter of a cylinder taken at one place, so that variations in the cylinder do not come into consideration.

Measurements	Deviation from Mean	
9.943 mm	0.0003	mm
9.942	- 0.0007	
9.944	0.0013	
9.941	- 0.0017	
9.943	0.0003	
9.943	0.0003	
9.945	0.0023	
9.943	0.0003	
9.941	- 0.0017	
9.942	- 0.0007	

How should one state the results of these measurements in a scientific fashion?

Expressed algebraically, the **mean** $\bar{x}$ of a set of n measurements $\{x_i\}$ is $\bar{x} = \sum x_i/n$. The mean of the 10 readings above turns out to be 9.9427 mm.

Column 2 gives each reading's deviation from the mean. The **standard deviation** σ (also known as the “root mean square deviation”) is defined as

$$\sigma = \text{standard deviation} = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2}$$

The standard deviation of the above 10 measurements is 1.225×10^{-3} .

Note that if you make one measurement of some quantity x , then $n = 1$ and $\bar{x} = x_1$ in the above equation: the standard deviation of your measurement is $\sqrt{0/0}$, which is undefined. You don't know how reliable your measurement is.

Assuming you make more than one measurement, σ tells you the typical deviation from the mean you will find for an individual measurement. It depends on your experimental method. If you make more measurements, σ will not change much.

However, the more measurements you make, the more accurately you can expect to know the value of the mean. The uncertainty of the value of the mean is called **the standard deviation of the mean** σ_μ , and in general it is less than σ :

$$\sigma_\mu = \text{standard deviation of the mean} = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

The standard deviation of the mean of the above 10 measurements is 3.96×10^{-4} .

When you take more measurements in order to minimize σ_μ , you are “gathering statistics”.

One states one's results as **mean $\pm$ standard deviation of the mean**. From the data above, the diameter of the cylinder was measured to be 9.9427 ± 0.0004 mm.

Error Distribution and Error Bars

For many types of measurement one may assume that the error distribution is “normal”: the probability of a given error ϵ is proportional to $e^{-\epsilon^2}$. If the error distribution is normal and the number of measurements is large, 68% of the measurements will lie closer than σ to the true value.

The purpose of the error bars shown on a graph in a technical report is as follows: if the reader attempts to reproduce the results in the graph using the procedure described in the report, the reader should expect his or her results to have a 50% chance of falling within the range indicated by the error bars.

If the error distribution is normal, the error bars should be of length $\pm 0.6745\sigma$.

Propagation of Uncertainties

Let $R = f(x, y, z)$ depend on measurements of three different quantities x , y , and z . The uncertainty ΔR in R which results from an uncertainty Δx in the measurement of x is

$$\Delta R = \frac{\partial f}{\partial x} \Delta x ,$$

and the fractional uncertainty in R is

$$\frac{\Delta R}{R} = \frac{\frac{\partial f}{\partial x}}{f} \Delta x .$$

In many experimental situations, the errors are uncorrelated and normally distributed. In this case the uncertainties add in quadrature (the square root of the sum of the squares):

$$\frac{\Delta R}{R} = \sqrt{\left(\frac{\frac{\partial f}{\partial x}}{f} \Delta x\right)^2 + \left(\frac{\frac{\partial f}{\partial y}}{f} \Delta y\right)^2 + \left(\frac{\frac{\partial f}{\partial z}}{f} \Delta z\right)^2} .$$

Δ denotes the uncertainty in a quantity, and σ refers to the standard deviation.

Some examples:

A) $R = x + y$. If the uncertainties in x and y each have a normal distribution and are independent, they combine in quadrature:

$$\Delta R = \sqrt{\Delta x^2 + \Delta y^2} .$$

Note that if $R = x - y$, then $\Delta R/R$ can become very large if x is nearly equal to y . Hence avoid, if possible, designing an experiment where one measures two large quantities and takes their difference to obtain the desired quantity.

B) $R = xy$. The fractional uncertainties combine in quadrature:

$$\frac{\Delta R}{R} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2} .$$

Note the same result occurs for $R = x/y$.

C) Consider the density ρ of a solid cylinder (Exp. M-201b):

$$\rho = \frac{m}{\pi r^2 L}$$

where m = mass, r = radius, L = length, are the three measured quantities. Then

$$\frac{\partial \rho}{\partial m} = \frac{1}{\pi r^2 L} \quad \frac{\partial \rho}{\partial r} = \frac{-2m}{\pi r^3 L} \quad \frac{\partial \rho}{\partial L} = \frac{-m}{\pi r^2 L^2} .$$

If the errors have normal distribution and are uncorrelated, then

$$\frac{\Delta \rho}{\rho} = \sqrt{\left(\frac{\Delta m}{m}\right)^2 + \left(2\frac{\Delta r}{r}\right)^2 + \left(\frac{\Delta L}{L}\right)^2}$$

D) Area of a circle: $A = \pi d^2/4$: $\Delta A/A = 2\Delta d/d$.

Significant Figures

Suppose you measure the diameter of a circular disk to be 24.326 ± 0.003 mm and want to compute its area $A = \pi d^2/4$. Evidently $A = 464.7627\ldots$ mm², but how many digits are significant?

The fractional uncertainty in d is $\Delta d/d = \frac{0.003}{24.326} = 1.2 \times 10^{-4}$. From example D) on the last page, this gives $\Delta A/A = 2.4 \times 10^{-4}$, so the uncertainty in A is $\Delta A = .11$ mm².

This tells us that only the first five digits are significant, so A should be reported as

$$A = \pi d^2/4 = 464.77 \pm 0.11 \text{ mm}^2$$

A good rule is to use one more digit in constants than is available in your measurements, and to save one more digit in computations than the number of significant figures in the data. When you use a calculator you usually get many more digits than you need. Therefore **at the end**, be sure to **round off** the final answer to display the correct number of **significant figures**.

SAMPLE QUESTIONS

- You measure $d = 1.0 \pm 0.1$ cm and $s = 1.0 \pm 0.1$ cm. What is the uncertainty in $r = d + s$?
 - 0.1 cm
 - 0.14 cm
 - 0.2 cm
- You measure r with a percentage uncertainty of 1%. What is the percentage uncertainty in $A = \pi r^2$?
 - 1%
 - 1.4%
 - 2%
- What happens to σ , the standard deviation, as you make more and more measurements? What happens to σ_μ , the standard deviation of the mean?
 - They both remain same
 - They both decrease
 - σ increases and σ_μ decreases
 - σ approaches a constant and σ_μ decreases
- You measure $m_1 = 115.34 \pm 0.01$ g and $m_2 = 1.0 \pm 0.1$ kg, in order to find $m_3 = m_1 + m_2$. How should you report m_3 ?
 - 1.11534 ± 0.00001 kg
 - 1.11534 ± 0.1 kg
 - 1.1 ± 0.1 kg